SOME INCOMPLETE NOTES ON ULTRAPRODUCT, FRAÏSSÉ LIMITS, ETC

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1. Introduction

Informally, let $\mathcal{K}$ be a class of structures (graphs, groups, hypergraphs, rings...). Our main questions of interest for this course are the following:

Question 1.1. Does there exists an object $M_{\mathcal{K}}$ which encodes the complexity of the class $\mathcal{K}$? If so, how do we construct it? What properties does this structure have? If such a *perfect* structure does not exist, are there alternatives? In which ways can we construct an object from the class $\mathcal{K}$ which *resembles* $\mathcal{K}$?

In general, there are several different solution to this problem. In this course, we will study three solutions:

(1) Fraïssé limits.
(2) Ultraproducts
(3) $\{0,1\}$-laws.

We give a crude description of each of these resolutions to our question:

(1) If $\mathcal{K}$ has several nice properties (i.e., Hereditary products, Joint Embedding property, and the amalgamation property), then the Fraïssé limit exists. This object really encodes the complexity from the class and is unique. Moreover, from the Fraïssé limit, one can recover the original class by considering the so called *age* of the limit.
(2) If $\mathcal{K}$ is arbitrary, one can always take an ultraproduct of the class. Ultraproducts are a kind of *forced convergence*. They will always exist. However, they may not reflect the complexity of the entire class – generally it concentrates on a subsection of the class and it dependent on the choice of ultrafilter. This constructions arises in many areas of mathematics [from combinatorics to Banach space theory] and is extremely useful for building counterexamples.
(3) One can think of probabalistic limits (or $\{0,1\}$-laws) as somewhere in-between Fraïssé limits and ultraproducts. The limit in this case is known as the *almost sure* theory. Like Fraïssé limits, it does not always exist, but when it does exist, it describes the *eventual theory* of arbitrarily large structures from the class.

2. Frassïssé limits and the random graph

Throughout this section, we let $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures. We to describe an object which encodes the complexity of $\mathcal{K}$.

Definition 2.1. We say that M is $\mathcal{K}$-universal if for every $A \in \mathcal{K}$, there exists an embedding from A into M.

Definition 2.2. We say that M is ultrahomogenous if for any two finitely generated substructures A, B from M, this there exists an automorphism $f : A \to B$, then there exists an automorphism $\alpha : M \to M$ such that $\alpha|_A = f$.

Definition 2.3. We let the age of M, age(M), be the class of structures which are isomorphic to a finitely generated substructure of M.

Definition 2.4. We say that M is a Fraïssé limit of the class $\mathcal{K}$ if M is countable, ultrahomogenous, $\mathcal{K}$-universal, and age$(M) = \mathcal{K}$.

2.1. **Finite graphs.** An important example is in the case of finite graphs. Recall that a graph $G = (V, E)$ is a collection of vertices V and a collection of edges $E \subseteq V^2 \setminus \{(v,v) : v \in V\}$ which is symmetric. Another presentation is as follows: a graph G is a structure in the language $\mathcal{L} = \{R(x,y)\}$ where R is a single binary relation and

(1) $G \models \forall x \forall y (R(x,y) \to R(y,x))$.
(2) $G \models \forall x \neg R(x,x)$.

Elements of G are are vertices; $G \models R(a,b)$ if and only if there is an edge between a and b.

Question 2.5. Does $\mathcal{K}$ admit a Fraïssé limit?

Try: Give an example of a $\mathcal{K}$-universal graph.

Let $\mathcal{K}(\mathbb{N}) = \{G$ is a finite graph: vertex set of G is a subset of $\mathbb{N}\}$. Consider $M = \bigsqcup_{G \in \mathcal{K}(\mathbb{N})} G$ were $M \models R(a,b)$ if and only if a anb b come from the same graph G and $G \models R(a,b)$.

It is obvious that M is $\mathcal{K}$-universal. It is easy to check that M is not ultrahomogenous.

Try 2: We build the Rado graph (also known as the Random graph), N.

Step 0: Let $N_0 = \{0\}$.

Step $k+1$: Suppose we have constructed N_k. We let $N_{k+1} := N_k \sqcup \{v_A^{k+1} : A \subseteq \mathcal{P}(N_k)\}$. So, we have the graph N_k and we add $2^{|N_k|}$-many new vertices. We an edge between an element a of N_k and a new vertex v_B^{k+1} if and only if $a \in B$. Otherwise, we do not add any new edges and we do not add any edges between pairs of new vertices.

Step ω: Consider the graph $\bigcup_{k \in \mathbb{N}} N_k$.

Proposition 2.6. *For every $n, m \geq 0$, $N \models \forall x_1, ..., x_n \forall y_1, ..., y_n \exists z \left(\bigwedge_{i \leq n}^{n} R(x_i, z) \wedge \bigwedge_{j \leq m} \neg R(y_j, z) \right)$.*

Proof. Fix $a_1, ..., a_n, b_1, ..., b_m \in N$. Let t be the smallest number such that $a_1, ..., a_n, b_1, ..., b_m \in N_t$. Then the appropriate element/vertex is found in N_{t+1}. ☐

Proposition 2.7. *For any finite graph H, there exists some $A \subseteq N$ such that A with the induced structure is isomorphic to H.*

Proof. By induction on the size of H. Suppose that the statement is true for graphs of size $n-1$. Suppose $|H| = n$ and let $H' \subseteq H$ such that $|H'| = n-1$. Let $H \setminus H' = b$. Let $B = \{a \in H' : H \models R(a,b)\}$. By the induction hypothesis, there exists some $A' \subseteq N$ such that $H' \cong A'$. Let t be the smallest number such that $A' \subseteq N_t$. We claim that $A' \cup v_B^{t+1} \cong H$. ☐

Corollary 2.8. *N is $\mathcal{K}$-universal.*

Question 2.9. Is N also ultrahomogenous?

Yes.

Proposition 2.10. *Suppose that $A, B \subseteq N$ such that $f : A \to B$ is a graph isomorphism. Then $\exists \alpha : N \to N$ an automorphism such that $\alpha|_A = f$.*

Proof. Enumerate two copies of N.

(1) $\underbrace{a_1, ..., a_n}_{A}, a_{n+1} ....$
(2) $\underbrace{b_1, ..., b_n}_{B}, b_{n+1}, ...$

We build our automorphism in stages.

Step 0: Let $\alpha_0 = f$.

Step $k+1.i$: Suppose we have constructed α_k with domain A_k and image B_k. Let t be the smallest index such that $a_t \notin A_k$. Let $C = \{a \in A_k : N \models R(a, a_t)\}$. It follows by Proposition 2.6 that there exists some b_* such that $b_* \notin B_K$ and for all $b \in B_k$, $N \models R(b_*, b)$ if and only if $b \in \alpha_k(C)$. Let s be the smallest index such that $b_s \notin B_k$ and $b \in B_k$, $N \models R(b_s, b)$ if and only if $b \in \alpha_k(C)$. Set $\alpha'_{k+1} = \alpha_k \cup \{(a_t, b_s)\}$.

Step $k+1.ii$: [To entail surjectivity, we need to go back.] Let A'_k be the domain of α'_{k+1} and the image be B'_k. Let t' be the smallest index such that $b_{t'}$ is not in B'_k.[A similar argument allows one to find a corresponding element in the first enumeration and send it to b [1].

We let $\alpha = \bigcup_{k\in\mathbb{N}} \alpha_k$. We claim that α is an isomorphism. □

Theorem 2.11. *N is a Fraïssé limit of the class of all finite graphs.*

3. First-order logic

First-order logic is a general framework in which to study arbitrary structures and/or classes of structures. The structures in question usually have kind of algebraic flavor (groups, rings, fields, graphs). In recent years, there has been a push to work with a variant of first order logic called *continuous logic* which allows one to appropriately talk about continuous objects (e.g., Hilbert spaces, measure algebras).

Another important reason why we care about first-order structures is because they formalize a collection of sentences which can be true about any particular strucutre. Hence, we can formally discuss *all* the first-order sentences which are true about a structure M.

Formally, a first order language $\mathcal{L}$ is a collection of function symbols, relation symbols, and constant symbols. The relation symbols and function symbols come equipped with a *fixed arity.* These means that a language is something like $\mathcal{L} = \{R(x,y), f(x)\}$ where R is a binary relation and f is a unary function. Or $\mathcal{L} = \{+, \times, 0, 1\}$ where $+, \times$ are binary function symbols and $0, 1$ are constants.

An $\mathcal{L}$-sentence is a *coherent* string of symbols below.

1. Logical symbols, all languages have the following:
   (a) '(' and ')'.
   (b) Connectives, $(\to, \wedge, \vee, \neg)$.
   (c) Variables $(v_i)_{i\in\mathbb{N}}$ (In formal proofs, we have this countable of variables. In practice, we usually use the symbols $x, y, z...$).
   (d) An equality symbols '='.
   (e) Quantifiers $\forall, \exists$.
2. Most importantly, symbols from $\mathcal{L}$
   (a) A collection of function symbols (each with fixed arity). This can be possibly empty.
   (b) A collection of Relation symbols (each with fixed arity). This can be possible empty.
   (c) A collection of constant symbols. This can be possibly empty.

In practice, we may write $\mathcal{L} = \{(f_i)_{i\in I}, (R_j)_{j\in J}, (c_k)_{k\in K}\}$ where the f_i's are function symbols, the R_j's are relation symbols, and the c_k's are constant symbols.

Remark 3.1. In an $\mathcal{L}$-sentence, all of the variables are bound. For example, if $\mathcal{L} = \{+\}$ then $\forall x \forall y(x+y = y+x)$ is a sentence while $\forall y(x+y = y+x)$ is not a sentence.

A coherent string of symbols with free variables such as $\forall y(x+y = y+x)$ is called an $\mathcal{L}$-formula. All $\mathcal{L}$-sentences are $\mathcal{L}$-formulas.

4. Models and satisfaction

An $\mathcal{L}$-structure is an object in which $\mathcal{L}$-sentences can be true or false with respect to that particular structure.

Definition 4.1. Let $\mathcal{L} = \{f_1, ..., f_n, R_1, ..., R_m, c_1, ..., c_k\}$. Then an $\mathcal{L}$-structure (also called an $\mathcal{L}$-model) is a tuple $(A; f_1^M, ..., R_1^M, ..., c_1^M)$ where

1. A is a non-empty set.
2. An interpreation for each function, relation, and constant symbol.
   (a) For each n-ary function symbol f_i in $\mathcal{L}$, $f_i^M : A^n \to A$.
   (b) For each n-ary relation symbol R_i in $\mathcal{L}$, $R_i^M \subseteq A^n$.
   (c) For each constant symbol c_i in $\mathcal{L}$, $c_i^M \in A$.

[1]Since these are notes, I will be a little pedantic. Let $D = \{b \in B'_k : N \models N(b, b_{t'})\}$. Again, by Proposition 2.6, there exists some a_* such that $a_* \notin A'_k$ and for all $a \in A'_k$, $N \models R(a_*, a)$ if and only if $a \in (\alpha_{k+1})^{-1}(D)$. Let s' be the smallest index such that $a_{s'}$ satisfies the above condition. Set $\alpha_{k+1} = \alpha'_{k+1} \cup \{(a_{s'}, b_{t'})\}$

The point is for each symbol from the language, we need to give an interpretation in our structure.

Example 4.2. Consider the language $\mathcal{L} = \{R(x, y), f(x)\}$ where R is a binary relation symbol and $f(x)$ is a unary function symbol. Here are some example of structures:

(1) $M = (\mathbb{Z}; <, S)$ where $R =<$ and $f = S$, the successor function, $n \to n + 1$.
(2) $N = (\mathbb{N}; x|y, x \to x^2)$ where R is interpretted as 'x divides y' and f is interpreted as the squaring function.
(3) $N' = (\{1, 2, 3, 4, 5\}; R, f)$ where R is the binary relation which holds only on $\{(1, 2), (3, 3)\}$ and f is the function which sends every to 1.

Definition 4.3. We write $M \models \varphi$ if φ is true relative to M.

Remark 4.4. For every $\mathcal{L}$-sentence, φ is either true or false with respect to M; so either $M \models \varphi$ or $M \models \neg\varphi$. It cannot $\models$ both φ and $\neg\varphi$.

Example 4.5. Consider the structure $M = (\mathbb{Z}; +, \times, 0, 1)$. Then

$$M \models \forall x \exists y (x + y = 0),$$

but

$$M \not\models \exists x \forall y (x + y = 0).$$

Definition 4.6 (Definable set)**.** Let $M = (A; ...)$ be an $\mathcal{L}$-structure. Let $D \subseteq A^n$. We say that D is definable if there exists an $\mathcal{L}$-formula $\varphi(x_1, ..., x_n)$ such that $(a_1, ..., a_n) \in D$ if and only if $M \models \varphi(a_1, ..., a_n)$.

Example 4.7. Consider $(\mathbb{N}; +, \times, 0, 1)$.

(1) Is $\{0\}$ a definable subset of $\mathbb{N}$? Yes. Consider the formula
$$\varphi_0(x) := \forall y (x + y = y + x).$$
(2) Is $\{1\}$ a definable subset of $\mathbb{N}$? Yes. Consider the formula
$$\varphi_1(x) := \forall y (x \times y = y \times x).$$
(3) Is $\{0, 1\}$ a definable subset of $\mathbb{N}$? Yes. Consider the formula
$$\varphi_{0,1}(x) := \varphi_0(x) \vee \varphi_1(x).$$
(4) Is $\{n\}$ a definable subset of $\mathbb{N}$? Yes. Consider the formula
$$\varphi_n(x) := \exists y (\varphi_1(y) \wedge x = \underbrace{y + ... + y}_{n-times})$$
(5) Primes? Yes. Exercise.
(6) $\{(a, b) : a < b\}$? Yes. Exercise.

Definition 4.8. Let $M_1 = (A_1; ...)$ and $M_2 = (A_2; ...)$ be $\mathcal{L}$-structures. We say that a map $G : A_1 \to A_2$ is an embedding if

(1) $G : A_1 \to A_2$ is an injection.
(2) G preserves functions, relation and constant symbols, i.e.
   (a) For each n-ary function f_i in $\mathcal{L}$ and tuple $(a_1, ..., a_n) \in A_1^n$,
   $$G(f_i^{M_1}(a_1, ..., a_n)) = f_i^{M_2}(G(a_1), ..., G(a_n)).$$
   (b) For each n-ary relation symbol R_i in $\mathcal{L}$ and tuple $(a_1, ..., a_n) \in A_1^n$, $(a_1, ..., a_n) \in R_i^{M_1}$ if and only if $(G(a_1), ..., G(a_n)) \in R_i^{M_2}$. In other words,
   $$M_1 \models R(a_1, ..., a_n) \iff M_2 \models R(G(a_1), ..., G(a_n)).$$
   (c) For each constant symbol c, $G(c^{M_1}) = c^{M_2}$.

We say that G is an isomorphism is G is also surjective. An automorphism is an isomrophism from M_1 to M_1.

For convention, we sometimes say an isomorphism G maps from M_1 to M_2.

Definition 4.9. Fix $\mathcal{L}$ and let $M = (A; ...)$ be an $\mathcal{L}$-structure. A map $G : A \to A$ is called an automorphism if it is an isomorphism.

Proposition 4.10. *Let $M = (A; ...)$ and $G : M \to M$ be an automorphism. Let $D \subseteq A^n$ be a definable set. Then for any $(a_1, ..., a_n) \in A^n$, we have that $(a_1, ..., a_n) \in D$ if and only if $(G(a_1), ..., G(a_n)) \in D$.*

The previous result allows us to show that certain subsets of a first order structures are not definable.

Example 4.11. Consider $(\mathbb{Z}, S)$ where S is the usual successor function. Let $\mathbb{E} = \{n \in \mathbb{Z} : n \text{ is even}\}$. Consider the map $\sigma : \mathbb{Z} \to \mathbb{Z}$ via $\sigma(n) = n + 1$. We claim that σ is an automorphism. Notice that if $\mathbb{E}$ where definable, then for any $a \in \mathbb{E}$, we have that $\sigma(a) \in \mathbb{E}$. However, $2 \in \mathbb{E}$, $\sigma(2) = 3$, and $3 \notin \mathbb{E}$. Hence $\mathbb{E}$ is not definable in this structure.

Example 4.12. Using the previous fact, one can show that $\{(a, b) : a < b\}$ is not a definable subset of $(\mathbb{R}; +, 0)$. However, it is a definable subset of $(\mathbb{R}; +, \times, 0, 1)$.

5. $\mathcal{L}$-theories and basic model theory

Definition 5.1. An $\mathcal{L}$-theory is a collection of $\mathcal{L}$-sentences. We say that a theory T is complete iff for every $\mathcal{L}$-sentence φ, either $\varphi \in T$ or $\neg\varphi \in T$. We say that T is satisfiable iff there exists some M such that $\forall \varphi \in T$, $M \models \varphi$, or in other words, $M \models T$.

Example 5.2. Let $\mathcal{L} = \{\leq\}$. Consider the following sentences:

(1) $\varphi_1 = \forall x(x \leq x)$.
(2) $\varphi_2 = \forall x \forall y \forall z((x \leq y \wedge y \leq z) \to x \leq z)$.
(3) $\varphi_3 = \forall x \forall y((x \leq y \wedge y \leq x) \to x = y)$.
(4) $\varphi_4 = \forall x \forall y(x \leq y \vee y \leq x)$
(5) $\varphi_5 = \forall x \exists y(x \neq y \wedge x \leq y)$.
(6) $\varphi_6 = \forall x \exists y(x \neq y \wedge y \leq x)$.
(7) $\varphi_7 = \forall x \forall y(x \neq y \wedge x \leq y \to \exists z(x \leq z \wedge z \leq y \wedge x \neq z \wedge y \neq z))$.

Let $\Sigma = \{\varphi_1, ..., \varphi_7\}$ is a theory.

Example 5.3. Let $\mathcal{L} = \{\emptyset\}$. Consider

$$\varphi_n = \exists x_1 ... \exists x_n \left(\bigwedge_{1 \leq i,j \leq n; i \neq j} x_i \neq x_j \right)$$

Then $\Sigma = \{\varphi_n : n \geq 1\}$ is a theory.

Example 5.4. If M is an $\mathcal{L}$-structure, then $Th_{\mathcal{L}}(M) = \{\varphi : M \models \varphi\}$ is both complete and satisfiable.

Theorem 5.5 (Compactness theorem)**.** *Let T be an $\mathcal{L}$-theory. Then Σ is satisfiable if and only if Σ is finitely satisfiable. In other words, there exists some M such that $M \models \Sigma$ if and only if for every $\Sigma_0 \subset \Sigma$ such that Σ_0 is finite, there exists some $M_0 \models \Sigma_0$.*

Example 5.6. Using the compactness theorem, we can build non-standard models of arithmetic. Consider the language $\mathcal{L} = \{+, \times, 0, 1, c\}$. We build a non-standard model of arithmetic. Consider the collection of sentence:

$$T = Th_{\mathcal{L}}(\mathbb{N}) \cup \{c > \underbrace{1 + ... + 1}_{n-times} : n \in \mathbb{N}\}$$

Using the compactness theorem, we can show that T is finitely satisfiable and thus T itself is satisfiable. Thus it has a model, N. Forgetting about the constant symbol gives a non-standard model of arithmetic. This structure, for example, has "infinite primes".

Theorem 5.7 (Upward-downward Lowenheim-Skolem theorem)**.** *Suppose that $|\mathcal{L}| \leq \aleph_0$ and T is a satisfiable theory with infinite models. Then for every infinite cardinal κ, there exists a model of T of size κ.*

Definition 5.8. Let M_1 and M_2 be $\mathcal{L}$-structures. We say that M_1 is *elementary equivalent* to M_2 and write $M_1 \equiv M_2$ if for every $\mathcal{L}$-sentences φ, $M_1 \models \varphi$ if and only if $M_2 \models \varphi$.

Proposition 5.9. *Let M_1 and M_2 be $\mathcal{L}$-structures. If $M_1 \cong M_2$, then $M_1 \equiv M_2$.*

Proof. Exercise. $\square$

Example 5.10. $(\mathbb{R}; \leq)$ and $(\mathbb{Q}; \leq)$ are not isomorphic, but they are elementary equivalent.

Question 5.11. How does one show this?

We describe a method which can work some of the time. Suppose that we are given two $\mathcal{L}$-structures, say N and M where M is countable. Then one may try to do the following

(1) Step 1: First, find a theory $T_0 \subseteq Th_{\mathcal{L}}(M)$ that sufficently captures the theory of M. Show that for every other countable $\mathcal{L}$-structure, if $N_0 \models T_0$, then $N_0 \cong M$.
(2) Step 2: Show that $N \models T_0$.

In the case of $(\mathbb{Q}; \leq)$ and $(\mathbb{R}; \leq)$, write down the axioms of a *dense linear ordering without endpoints*. Every countable model is isomorphic to $(\mathbb{Q}, \leq)$ [using a back-and-forth] argument. Obviously, $(\mathbb{R}; \leq)$ is also a dense linear order without endpoints.

6. Fraïssé's Theorem

Throughout this section, we assume that $|\mathcal{L}| = \aleph_0$. Let $\mathcal{K}$ be a class of $\mathcal{L}$ structures which is closed under isomorphism.

Definition 6.1. We say that $\mathcal{K}$ has the hereditary property (HP) if for every $A \in \mathcal{K}$, $\text{age}(A) \subseteq \mathcal{K}$.

Definition 6.2. We say that $\mathcal{K}$ has the joint embedding property (JEP) if for every $A, B \in \mathcal{K}$, there exists some $C \in \mathcal{K}$ such that A and B embed into C.

Definition 6.3. We say that $\mathcal{K}$ has the amalgamation property (AP) if for every $A, B_1, B_2 \in \mathcal{K}$ and embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$, there exists some $C \in \mathcal{K}$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that the diagram commutes; i.e., $g_1 \circ f_1 = g_2 \circ f_2$.

Proposition 6.4. *Let $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures such that $|\mathcal{K}/\cong|$ is countable. Then the following are equivalent:*

(1) There exists a countable $\mathcal{L}$-structure M such that $\text{age}(M) = \mathcal{K}$.
(2) $\mathcal{K}$ has the JEP and HP.

Proof. Straightforward. (2) $\implies$ (1), enumerate representatives of each isomorphism type of structure, say $(E_i)_{i<\omega}$.

Step 0: Let $D_0 = E_0$.

Step n+ 1: Suppose we have constructed D_n where $D_0 \subseteq ... \subseteq D_n$. Then use the joint embedding property to find a structure D_{n+1} so that E_n and D_n embed into it.

Let $M = \bigcup_{n<\omega} D_n$. $\square$

Remark 6.5. For example, let $\mathcal{K}$ be the class of finite linear orders. Then $(\mathbb{N}, <)$, $(\mathbb{Z}, <)$, and $(\mathbb{Q}, <)$ are structures such that their are is equal to $\mathcal{K}$.

Proposition 6.6. *Let $\mathcal{K}$ be the age of M and suppose that $|M| = \aleph_0$. Then the following are equivalent:*

(1) M is ultrahomogenous.
(2) For all $A, B \in \mathcal{K}$, if $f : A \to M$ and $g : A \to B$ are embeddings, then there exists some $h : B \to M$ such that the diagram commutes.
(3) For every finitely generated substructure A of M and $B \in \mathcal{K}$, if $f : A \to B$ is an embedding, the there exists a $g : B \to M$ such that $g \circ f = id_A$.

Proof. We briefly describe how the proofs work:

(*i*) (1) $\to$ (3) Diagram chase.
(*ii*) (3) $\to$ (1) A back-and-forth style argument.
(*iii*) (2) $\longleftrightarrow$ (3) is straightforward.

$\square$

Theorem 6.7 (Fraïssé's Theorem)**.** *Let $\mathcal{L}$ be a countable language. Let $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures which is closed under isomorphisms and has only countable many isomorphism classes, i.e., $|\mathcal{K}/\cong| = \aleph_0$. The the following are equivalent:*

(1) Then there exists a unique $\mathcal{L}$-structure M such that
  (a) $|M| = \aleph_0$.
  (b) $\text{age}(M) = \mathcal{K}$.
  (c) M is ultrahomogenous.

(2) $\mathcal{K}$ has the HP, JEP, and AP.

M is called the Fraïssé limit of the class $\mathcal{K}$.

Proof. (1) $\to$ (2); We will show that $\mathcal{K}$ has AP, the other conditions are straightforward. So suppose that A, B_1, B_2, f_1, f_2 are given. Since $\text{age}(M) = \mathcal{K}$, there exists a map $h : A \to M$. By Proposition 6.6, if we consider the map $f_1 \circ h^{-1} : h(A) \to B_1$, then there exists some map $g_1 : B_1 \to M$ such that $g_1 \circ f_1 \circ h^{-1} = id_{h(A)}$. Likewise, if we consider the map $f_2 \circ h^{-1} : h(A) \to B_2$, then there exists some map $g_2 : B_2 \to M$ such that $g_2 \circ f_2 \circ h^{-1} = id_{h(A)}$. Let $C = \langle g_1(B_1), g_2(B_2)\rangle$. We claim that the diagram with $A, B_1, B_2, f_1, f_2, g_1, g_2, C$ commutes and thus the amalgamation problem has been solved.

(2) $\to$ (1); This is a more complicated variant of Proposition 6.4. We will only check that the structure we build is ultrahomogenous. The other properties are either clear by construction or left to the reader as an exercise. We build a structure $M = \bigcup_{n<\omega} D_n$ such that

(1) For all $A \in \mathcal{K}$, A embeds into M.
(2) If $A, B \in \mathcal{K}$, $A \subseteq B$, and $f : A \to D_i$ is an embedding, then there exists some $j > i$ and an embedding $g : B \to D_j$ such that $g|_A = f$.

Notice that condition (2) above implies condition (3) from Proposition 6.6. We will construct a set of pairs of elements from $\mathcal{K}$. Indeed, $P \subseteq \{(A, B) : A, B \in \mathcal{K} \text{ and } A \subseteq B\}$ such that every possible isomorphism type of pairs has a representative. Let $\pi : \omega \times \omega \to \omega$ be a bijection such that $\pi(i, j) \geq i$. Let $(E_i)_{i<\omega}$ be an enumeration of representatives of the isomorphism types from $\mathcal{K}$.

Step 0**:** Let D_0 be any element from $\mathcal{K}$.

Step $k+1$**:** Suppose that we have constructed D_k and $D_0 \subseteq .... \subseteq D_k$. First, replace D_k with D'_k where we get D'_k via the JEP applied to D_k and E_k. We then add some labellings to the grid $\omega \times \omega$. We let $((f_{(k,j)}, A_{(k,j)}, B_{(k,j)}) : j < \omega)$ where this sequence is an enumeration of pairs from P along with all possible embeddings from A into D'_k. We remark that this enumeration is countable.

Then, if $\pi(i, j) = k$, we construct D_k as follows: Consider the elements A_{ij}, B_{ij}, f_{ij}. Then $f_{ij} : A_{ij} \to D_i \subseteq D_k \subseteq D'_k$ and $\iota : A_{ij} \to B_{ij}$ is the inclusion map. By AP, there exists some D_{k+1} completing the diagram. ☐

Example 6.8. The following classes of structures admit Fraïssé limits.

(1) Finite graphs; The Fraïssé limit is the random graph. It is axiomatized by the following collection of sentences:

$$\forall x_1, ..., x_n \forall y_1, ..., y_m \exists z \left(\bigwedge_{i=1}^{n} R(x_i, z) \wedge \bigwedge_{j=1}^{m} R(y_i, z) \right).$$

We know that this is the Fraïssé limit by Fraïssé's theorem, namely, there only exists one structure such that $\text{age}(M)$ is all finite graphs and M is ultrahomogenous. We proved previously that this structure had such properties.

(2) Finite linear orders; The Fraïssé limit is $(\mathbb{Q}, \leq)$.

7. Axiomizations

Fact 7.1. *Suppose that A is a finite $\mathcal{L}$-structure, say of size n. Then there exists a single $\mathcal{L}-$sentence φ such that if $B \models \varphi$, then $B \cong A$.*

The idea is to write out all of the atomic facts about the finite structure.

Example 7.2. Suppose that $\mathcal{L} = \{R(x,y)\}$ and G be a finite graph on the vertex set $\{1,...,n\}$. Then let

$$\varphi_G(\bar{x}) := \left(\bigwedge_{1\le i<j\le n} x_i \neq x_j \wedge \bigwedge_{\substack{1\le i<j\le n \\ G\models R(i,j)}} R(x_i,x_j) \wedge \bigwedge_{\substack{1\le i<j\le n \\ G\models \neg R(i,j)}} \neg R(x_i,x_j) \wedge \forall y \left(\bigvee_{i=1}^{n} y = x_i \right) \right)$$

Then if $H \models \exists \bar{x} \varphi_G(\bar{x})$, then $H \cong G$.

Recall: By definition, structures are non-empty.

Definition 7.3. Let T be an $\mathcal{L}$-theory. We say that T' is an axiomization of T if

(1) $T' \subseteq T$.
(2) If $M \models T'$ then $M \models T$.

Idea: T is general can be quite complicated; we want to reduce T to something less complicated which we can understand.

Example 7.4. Suppose that A is a finite $\mathcal{L}-$structure. Then $\{\exists \bar{x} \varphi_A(\bar{x})\}$ is an axiomization of $Th_{\mathcal{L}}(A)$.

Example 7.5. The theory of $(\mathbb{Q}, \le)$ is axiomatized by Σ from Example 5.2. This requires an argument; using back-and-forth.

We remark that in general, axiomizations are not unique.

Question 7.6. Is there a simple way to understand $Th_{\mathcal{L}}(M_{\mathcal{K}})$?

7.1. **Axiomizations of Fraïssé limits.** For this entire section, we assume that $\mathcal{L}$ is a finite relational language. Fix a Fraïssé class (of finite structures) $\mathcal{K}$. For each natural number n, we let $K(n)$ be the finite set of structures from $\mathcal{K}$ with domain $\{1,...,n\}$. Consider the following two families of sentences:

(1) For each natural number n;

$$\theta_n := \forall x_1, ..., x_n \left(\left(\bigwedge_{1\le i<j\le n} x_i \neq x_j \right) \to \left(\bigvee_{A\in K(n)} \varphi_A(\bar{x}) \right) \right).$$

(2) One point extension axiom: For all $A \in K(n)$ and $B \in K(n+1)$, we say that (A,B) is a one-point-extension if A is the induced substructure of B with domain $\{1,...,n\}$. Given a pair (A,B) which is a one-point-extension, we let

$$\psi_{A,B} := \forall \bar{x} \exists y \left(\varphi_A(\bar{x}) \to \varphi_B(\bar{x}, y) \right).$$

Lemma 7.7. *Suppose $\mathcal{L}$ is countable. Fix an $\mathcal{L}$-theory T and suppose that $M \models T$ where $|M| = \aleph_0$. Suppose that for every countable $\mathcal{L}$-structure N, if $N \models T$ then $N \cong M$. Then if $N' \models T$ we claim that $N' \equiv M$.*

Proof. Suppose not. Then there exists a uncountable $\mathcal{L}$-structure N' such that $N' \models T$, and there exists a sentence φ such that $M \models \varphi$ but $N' \models \neg\varphi$. By the downward Lowenhiem-skolem theorem, there exists a countable models $N_0 \models T \cup \{\neg\varphi\}$. But then $N_0 \cong M$ so $N_0 \equiv M$ and so $N_0 \models \varphi$. Thus $N_0 \models \varphi \wedge \neg\varphi$, a contradiction. ☐

Theorem 7.8. *Let $\mathcal{K}$ be a Fraïssé class in a finite relational language. Let $M_{\mathcal{K}}$ be the Fraïssé limit of $\mathcal{K}$ and $T = Th_{\mathcal{L}}(M_{\mathcal{K}})$. Then $T' = \{\theta_n, \psi_{A,B} : n \ge 1; (A,B)$ a 1-point-extension$\}$ is an axiomization of T.*

Proof. Back-and-forth. By Lemma 7.7, it suffices to prove claim 1 below:
Claim 1: Suppose that $|N| = \aleph_0$ and $N \models T'$. We claim that then $N \cong M_{\mathcal{K}}$, so $N \equiv M_{\mathcal{K}}$ and thus $N \models T$. We sketch the argument. Enumerate $M_{\mathcal{K}}$ by $c_1, c_2, ...$ and N by $d_1, d_2, ...$ Suppose that at step k, we have constructed the isomorphism $f_k : C_k \to D_k$ which are finite substructures of $M_{\mathcal{K}}$ and N respectively (say of size n). Let j be the smallest number such that c_j is not in the domain of f_k. Then $C_k \cup \{c_j\}$ is a substructure of $\mathcal{M}_{\mathcal{K}}$ (call it B'). Then

$$N \models \psi_{C_k \cup \{c_j\}, B'}$$

and in particular, we conclude that

$$N \models \varphi_{B'}(D_k, e)$$

where e is some d_l in the enumeration. Let $f_k' = f_k \cup \{(c_j, d_l)\}$. To complete the proof, one still needs to do the *back* portion of the back-and-forth. Then take a union of the constructed functions to get an isomorphism. □

8. Pseudofinite

8.1. Pseudofinite Fraïssé limits.

Definition 8.1. Let T be an $\mathcal{L}$-theory. We say that T is pseudofinite if for every finite subcollectino $T_0 \subseteq T$, there exists a finite model M_0 such that $M_0 \models T_0$.

Proposition 8.2. *Suppose that T' is an axiomization of T. If T' is pseudofinite, then T is pseudofinite.*

Proof. Exercise. □

Example 8.3. The random graph is pseuduofinite.

Proof. Consider

$$T' = \left\{ \forall x_1, ..., x_k \forall y_1, ..., y_\ell \left(\bigwedge_{i,j} x_i \neq y_j \to \exists z \left(\bigwedge_{i,j} zRx_i \wedge \neg zRy_j \right) \right) \right\}$$

The above collection of sentences axiomatizes the random graph [recall that there is no such thing as the empty-structure, by convention]. It suffices to show that each sentence above is true in a finite model (since $\varphi_{k,\ell}$ implies $\varphi_{k',\ell'}$ for any $k' \leq k$ and $\ell' \leq \ell$). The argument is probabilistic. For fixed $n \in \mathbb{N}$, consider $V = \{1, ..., n\}$. For each edge $e \in [V]^2$, we consider the probability space $\Omega_e = \{0_e, 1_e\}$ with $P_e(\{1_e\}) = t$. We let $G(n,t)$ be the probability space $\prod_{e \in [V]^2} \Omega_e$ with the product measure. Fix a pair of disjoint vertices U and W from v where $|U| \leq k$ and $|W| \leq \ell$. Then the probability that there is no suitable $v \in V$ which connects to every U and does not connect to any W is bounded by $(1 - t^k s^\ell)^{n-k-\ell}$ where $s = (1-t)$. Since there are no more than $n^{k+\ell}$ many pairs (U, W) with $U \cap W = \emptyset$ and $|U| \leq k$, $|W| \leq \ell$, we have that the probability that $\varphi_{k,\ell}$ fails for a random graph of size n is bounded by

$$n^{k+\ell} \cdot (1 - t^k s^\ell)^{n-k-\ell}$$

As $n \to \infty$, the above expression goes to 0. Thus, we may find such a graph which witnesses $\varphi_{k,\ell}$. □

Example 8.4. Let $\mathcal{K}$ be the class of all finite linear orders. Then we know that the theory of the Fraïssé limit is the same as the theory of $(\mathbb{Q}, \leq)$. This is not pseudofinite.

Question 8.5. Let $\mathcal{K}$ be the class of all finite triangle free graphs. It is still open whether or not the Fraïssé limit of this class is pseudofinite.

Question 8.6. When are Fraïssé limits pseudofinite?

A nice answer can be found in *Disjoint n-amalgamation and pseudofinite countably categorical theories* by Alex Kruckman - See `https://arxiv.org/abs/1510.03539`. We will simplify part of his analysis below just for our particular setting:

Definition 8.7. Let M be an $\mathcal{L}-$structure in a finite relational language. Then a quantifier free type (in variables $x_1, ..., x_n$ over $\emptyset$) is the collection of the relations [and negations] (including equality) which are true about a tuple of points $a_1, ..., a_n$ in M. Formally,

$$\begin{aligned} q(x_1, ..., x_n) &= \mathrm{tp}_{qf}(a_1, ..., a_n) \\ &= \{R(x_{i_1}, ..., x_{i_n}) : R \in \mathcal{L}, M \models R(a_{i_1}, ..., a_{i_n})\} \cup \{\neg R(x_{i_1}, ..., x_{i_n}) : R \in \mathcal{L}, M \models \neg R(a_{i_1}, ..., a_{i_n})\}. \end{aligned}$$

So, a quantifier free type (in variables $\{x_1, ..., x_n\}$ over $\emptyset$) is a *consistent* collection of data.

Definition 8.8. A basic quantifier free disjoint n-amalgamation problem is the following:

Take $\mathcal{P}([n])$ and let $\mathcal{P}^-([n]) = \mathcal{P}([n]) \setminus \{1, ..., n\}$. Consider the variables $x_1, ..., x_n$. Label each element S from $\mathcal{P}^-([n])$ with a quantifier free type (over the emptyset) in variables $\{x_i : i \in S\}$ such that if $S' \subseteq S$, then $p_S|_{\{x_i : i \in S'\}} = p_{S'}$. Does there exists a quantifier free type q in free variables $\{x_1, ..., x_n\}$ such that for all $S' \in \mathcal{P}^-([n])$, $q|_{\{x_i : i \in S'\}} = p_{S'}$? If such a q exists, q is called a solution.

We say that T has disjoint n-amalgamation if every n-amalgamation probelem has a solution.

Intuitively, can we consistently glue together quantifier free types.

Theorem 8.9. *Suppose that T has $n-$disjoint amalgamation for all n. Then T is pseudofinite.*

Proof. Probabilistic argument. See Theorem 3.10 in *Disjoint n-amalgamation and pseudofinite countably categorical theories.* □

9. Ultrafilters

Definition 9.1. Let I be an indexing set and $\mathcal{P}(I)$ denote the power set of I. A filter $\mathcal{F}$ (on I) is a non-empty subset of $\mathcal{P}(I)$ with the following properties:

(1) $\emptyset \notin \mathcal{F}$.
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(3) If $B \supseteq A$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

The following facts are easy to check.

Fact 9.2. *Let I be an indexing set and $\mathcal{F}$ is a filter on I.*

(1) For any finite collections $A_1, ..., A_n \in \mathcal{F}$, $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$.
(2) $I \in \mathcal{F}$.

Example 9.3. Let $I = \mathbb{N}$.

(1) For any $a \in \mathbb{N}$, we let $D_a = \{X \subseteq \mathbb{N} : a \in X\}$. D_a is a filter. Filters of this form are called *principal*.
(2) Let $\mathcal{F}_{cofinite} = \{X \subseteq \mathbb{N} : |\mathbb{N}\backslash X| < \aleph_0\}$. This is a filter and is known as the cofinite-filter or the *Frechet* filter.

Definition 9.4. Let I be an indexing set and let $\mathcal{F}$ be a filter on I. We say that F is an ultrafilter on I if for every $X \subseteq I$, either $X \in \mathcal{F}$ or $I\backslash X \in \mathcal{F}$.

Proposition 9.5. *Suppose that $\mathcal{F}$ is a filter on I. Suppose that $A \subset \mathbb{N}$ such that $A \notin \mathcal{F}$ and $I\backslash A \notin \mathcal{F}$. Let $\mathcal{F}_A = \{B \cap A : B \in \mathcal{F}\}$. Let $\overline{\mathcal{F}_A} = \{C : \exists E \in \mathcal{F}_A$ such that $C \supseteq E\}$. Then*

(1) $F \subset \mathcal{F}$.
(2) $\mathcal{F}$ is a filter on I.

Proof. Suppose that $C \in \mathcal{F}$. Then $C \supseteq C \cap A$ and so $C \in \mathcal{F}$. Moreover, $A \in \mathcal{F}$, but $A \notin F$ by assumption. Hence $F \subsetneq \mathcal{F}$.

We now show that $\mathcal{F}$ is a filter.

(1) Suppose that $\emptyset \in \mathcal{F}$. Then $\emptyset \in \mathcal{F}_A$. Hence there exists some $B \in \mathcal{F}$ such that $B \cap A = \emptyset$. Then $I\backslash A \supseteq B$ which implies that $I\backslash A \in F$. This contradicts our assumption.
(2) Suppose that $C_1, C_2 \in \mathcal{F}$. Then there exists $B_1, B_2 \in \mathcal{F}$ such that $C_1 \supseteq B_1 \cap A$ and $C_2 \supseteq B_2 \cap A$. Then $B_1 \cap B_2 \in \mathcal{F}$ and $C_1 \cap C_2 \supseteq (B_1 \cap B_2) \cap A$. By construction, $C_1 \cap C_2 \in \mathcal{F}$.
(3) Suppose that $C_1 \in \mathcal{F}$ and $C_2 \supseteq C_1$. Then there exists $B \in \mathcal{F}$ such that $C_1 \supseteq B \cap A$. So $C_2 \supseteq B \cap A$ and therefore $C_2 \in \mathcal{F}$.

Hence $\mathcal{F}$ is a filter. □

Theorem 9.6. *Let I be an indexing set and suppose that $\mathcal{F}$ is a filter on I. Then there exists an ultrafilter D on I such that $\mathcal{F} \subseteq D$.*

Proof. This follows from an application of Zorn's lemma. Consider $(\mathcal{G}, \subseteq)$ where $\mathcal{G} = \{D : D$ is a filter over I and $D \supseteq \mathcal{F}\}$. Let $(\mathcal{C}, \subseteq)$ be a chain in this partial order. We need to show that this chain has an upper bound. Consider the set $H = \bigcup_{D \in \mathcal{C}} D$. We claim that $H \in \mathcal{G}$ and H is an upper bound for C. It suffices to show that H is a filter.

(1) Suppose that $\emptyset \in H$. Then there exists some $D \in \mathcal{C}$ such that $\emptyset \in D$, but this is a contradiction since D is a filter. Hence $\emptyset \notin H$.
(2) Suppose that $A_1, A_2 \in H$. Then there exists some $D \in \mathcal{C}$ such that $A_1, A_2 \in D$. Then $A_1 \cap A_2 \in D$ which implies $A_1 \cap A_2 \in H$.
(3) Suppose that $A_1 \supseteq A_2$ and $A_2 \in H$. Then there is some $D \in \mathcal{C}$ such that $A_2 \in D$. So $A_1 \in D$ and so $A_1 \in H$.

By Zorn's lemma, there exists a maximal element $K \in \mathcal{G}$. Since $K \in \mathcal{G}$, we know that $F \subseteq K$. We claim that K is an ultrafilter. Assume not. Then there exists some $A \subseteq I$ such that $A \notin K$ and $I \backslash A \notin K$. By the previous proposition $\overline{K_A}$ is a filter which properly extends K. Thus K is not maximal and so we have a contradiction. ☐

Definition 9.7. Let I be an indexing set and D be an ultrafilter on I. We say that D is a principle ultrafilter if there exists some $i \in I$ such that $D = D_i = \{X \subseteq I : i \in X\}$. Otherwise, we say that D is non-principle.

Proposition 9.8. *Suppose that I is finite. Then every ultrafilter on I is principle.*

Proposition 9.9. *Suppose that I is infinite. Then there exists a non-principle ultrafilter on I. Moreover, if D is a non-principle ultrafilter on I, then D contains every cofinite set, i.e. for any $X \subseteq I$ such that $|I \backslash X| < \aleph_0$, $X \in D$.*

10. Ultraproducts

Definition 10.1. Let I be an indexing set and $(M_i)_{i \in I}$ an indexed family of $\mathcal{L}$-structures. We consider the product $\prod_{i \in I} M_i$. Notice that every element in $\prod_{i \in I} M_i$ can be thought of as a function $f : I \to \bigcup_{i \in I} M_i$ where $f(i) \in M_i$. Elements of $\prod_{i \in I} M_i$ can also be thought of as sequences of points $(a_1, a_2, a_3, ...)$ where each $a_i \in M_i$.

Now let D be a filter on I. We define a relation $\sim_D$ on $\prod_{i \in I} M_i$ where $f \sim_D g$ if and only if $\{i \in I : f(i) = g(i)\} \in D$.

Proposition 10.2. *Let I be an indexing set, $(M_i)_{i \in I}$ an indexed family of $\mathcal{L}$-structures, and D be a filter on I. Then $\sim_D$ is an equivalence relation on $\prod_{i \in I} M_i$.*

Proof. Exercise. ☐

Definition 10.3. Let I be an indexing set, $(M_i)_{i \in I}$ an indexed family of $\mathcal{L}$-structures, and D be an ultrafilter on I. We let $\prod_D M_i = \prod_{i \in I} M_i / \sim_D$. In other words, if $[f]_D = \{g \in \prod_{i \in I} M_i : f \sim_D g\}$, then $\prod_D M_i = \{[f]_D : f \in \prod_{i \in I} M_i\}$. $\prod_D M_i$ is an $\mathcal{L}$-structure with the following interpretations of $\mathcal{L}$-symbols (for ease of notation, we let $N = \prod_D M_i$):

(1) Let R be an n-ary relation symbol. Then $([f_1]_D, ..., [f_n]_D) \in R^N$ if and only if $\{i \in I : (f_1(i), ..., f_n(i)) \in R^{M_i}\} \in D$.
(2) Let G be an n-ary function symbols. Then $G^N([f_1]_D, ..., [f_n]_D) = [h]_D$ where $h(i) = G^{M_i}(f_1(i), ..., f_n(i))$.
(3) Let c be a constant symbol. Then $c^N = [f_c]_D$ where $f_c(i) = c^{M_i}$.

The structure $\prod_D M_i$ is called an ultraproduct.

Theorem 10.4 (Łoś's Theorem)**.** . *Let I be an indexing set, D an ultrafilter on I, $(M_i)_{i \in I}$ an indexed family of $\mathcal{L}$-structures, and $f_1, ..., f_n \in \prod_{i \in I} M_i$. Then for any $\mathcal{L}$-formula $\varphi(x_1, ..., x_n)$,*

$$\prod_D M_i \models \varphi([f_1]_D, ..., [f_n]_D) \iff \{i \in I : M_i \models \varphi(f_1(i), ..., f_n(i))\} \in D.$$

Moreover, for any $\mathcal{L}$-sentence φ, we have that

$$\prod_D M_i \models \varphi \iff \{i \in I : M_i \models \varphi\} \in D.$$

Proof. Induction hypothesis: Suppose the condition holds for $\psi(x_1, ..., x_n)$ and $\theta(x_1, ..., x_n)$.

Conjunction: Follows from intersection part.

$$\begin{aligned}
&\prod_D M_i \models \psi([f_1]_D, ..., [f_n]_D) \wedge \theta([f_1]_D, ..., [f_n]_D) \\
\iff &\prod_D M_i \models \psi([f_1]_D, ..., [f_n]_D) \text{ and } \prod_D M_i \models \theta([f_1]_D, ..., [f_n]_D) \\
\iff &\{i \in I : M_i \models \psi(f_1(i), ..., f_n(i))\} \in D \text{ and } \{i \in I : M_i \models \psi(f_1(i), ..., f_n(i))\} \in D \\
\iff &\{i \in I : M_i \models \psi(f_1(i), ..., f_n(i))\} \cap \{i \in I : M_i \models \psi(f_1(i), ..., f_n(i))\} \in D \\
\iff &\{i \in I : M_i \models \psi(f_1(i), ..., f_n(i)) \wedge \psi(f_1(i), ..., f_n(i))\} \in D
\end{aligned}$$

Negation: Check.

Existential quantifier: We want to show that the statement holds for $\exists x_1 \psi(x_1, ..., x_n)$.

$$\begin{aligned}
\prod_D M_i \models \exists x_1 \psi(x_1, [f_2]_D, ..., [f_n]_D) &\iff \prod_D M_i \models \psi([g]_D, [f_2]_D, ..., [f_n]_D) \\
&\iff \{i \in I : M_i \models \psi(g(i), f_2(i), ..., f_n(i))\} \in D \\
&\iff \{i \in I : M_i \models \exists x \psi(x, f_2(i), ..., f_n(i))\} \in D.
\end{aligned}$$

Last if and only if forward direction is trivial, backwards direction "relies on the Axiom of choice". □

Corollary 10.5. *Let $(M_i)_{i \in I}$ be an indexed family of $\mathcal{L}$-structures, M be an $\mathcal{L}$-structure, and suppose that $M \equiv M_i \equiv M_j$ for each $i, j \in I$. If D is an ultrafilter on I, then*

$$\prod_D M_i \equiv M.$$

Definition 10.6. Let I be an indexing set, $M_i = M$ for every $i \in I$, and D be an ultrafilter on I. Then $\prod_D M_i$ is called an ultrapower and there is a natural map $\Delta : M \to \prod_D M$ via $\Delta(a) = [(a_0, a_1, a_2, ...)]_D$. In other words, an element a in M is mapped to equivalence class of the constant function f_a where for any $i \in I$, $f_a(i) = a$.

Example 10.7. Let $I = \omega$, $M_i = (\mathbb{N}, <)$ for each $i < \omega$, and D be an ultrafilter on I which extends the cofinite filter. Then

(1) We notice that there are elements in $\prod_D M_i$ which are larger than every standard natural numbers, e.g. $[(0, 1, 2, 3, 4, ...)]_D$.
(2) $\prod_D M_i$ has no greatest elements since $\{i \in I : M_i \models \forall x \exists y (x < y)\} \in D$.
(3) We notice that $\prod_D M_i$ is not well-ordered: Consider the sequence

$$[(0, 1, 2, 3, 4, 5, ...)]_D > [(0, 0, 1, 2, 3, 4, ...)]_D > [(0, 0, 0, 1, 2, 3)]_D ...$$

Example 10.8. Consider the theory of algebraically closed fields of characteristic p (ACF_p) in the language $\mathcal{L}_{ring} = \{+, \times, 0, 1\}$. These theories say

(1) The structure is a field.
(2) The structure is algebraically closed (i.e., every polynomial has a solution).
(3) $\underbrace{1 + ... + 1}_{p-times} = 0$

For each prime p, we let $\mathbf{F}_p \models ACF_p$. Then $\prod_D \mathbf{F}_p \models ACF_0$. More generically, one can prove that $\prod_D \mathbf{F}_p \equiv (\mathbb{C}; +, \times, 0, 1)$.

The following lemma is quite helpful.

Lemma 10.9. *Let T be a theory. Then I be the collection of finite subsets of T. Suppose that for every $i \in I$, there exists some M_i such that $M_i \models i$. Then there exists an ultrafilter D on I such that $\prod_D M_i \models T$.*

Proof. For each $\varphi \in T$ we let $\hat{\varphi} = \{i \in I : \varphi \in i\}$. Consider the set $E = \{\hat{\varphi} : \varphi \in T\}$. This set have the finite intersection property. In other words, if we find $\hat{\varphi}_1, ..., \hat{\varphi}_n \in E$, then $\bigcap_{i=1}^n \hat{\varphi}_i \neq \emptyset$. In particular, notice that $\{\varphi_1, ..., \varphi_n\}$ is an element of the intersection. One can prove that there exists an ultrafilter D such that $E \subseteq D$ [extend E to a filter, then extend to an ultrafilter using Zorn's lemma]. We claim that $\prod_D M_i \models T$. Suppose that $\varphi \in T$. Then $\hat{\varphi} \in E$. Then $\{i \in I : M_i \models \varphi\} \subseteq \hat{\varphi} \in D$. Hence $\prod_D M_i \models \varphi$. □

Remark 10.10. One can prove the compactness theorem directly from Los's theorem using the lemma above.

Corollary 10.11. *Let T be a theory. Then T is pseudofinite if and only if there exists an indexing set I, an indexed family of finite structures $(M_i)_{i \in I}$, and an ultrafilter D on I such that $\prod_D M_i \models T$.*

Proof. Direct from 10.9. □

11. Elementary Classes

Definition 11.1. Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. We say that $\mathcal{K}$ if an elementary class if and only if there exists an $\mathcal{L}$-theory T such that

$$M \in \mathcal{K} \iff M \models T.$$

Moreover, we say that an elementary class is a basic elementary class if one can choose the theory T above such that it is finite.

Proposition 11.2. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. Then $\mathcal{K}$ is an elementary class if and only if $\mathcal{K}$ is closed under elementary equivalence and ultraproducts.*

Proof. The forward direction follows from Los's theorem. We now show the backwards direction. Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures which is closed under elementary equivalence and ultraproducts. Consider the theory T where

$$T := \{\varphi : \forall M \in \mathcal{K}, M \models \varphi\}.$$

We show that T shows that $\mathcal{K}$ is elementary. If $M \in \mathcal{K}$, then clearly $M \models T$. On the other hand, suppose that $M \models T$. Let I be the collection of finite subsets of $Th_{\mathcal{L}}(M)$. Then for each $i \in I$, we can find some $N_0 \in \mathcal{K}$ such that $N_0 \models i$ [To see why this is true, suppose not... contradict the definition of T]. Then by Lemma 10.9, there exists an ultrafilter D on I such that $\prod_D N_i \equiv M$ where each $N_i \in \mathcal{K}$. Since $\mathcal{K}$ is closed under ultraproducts and elementary equivalence, we conclude that $M \in \mathcal{K}$. □

Proposition 11.3. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. Then $\mathcal{K}$ is a basic elementary class if and only if both $\mathcal{K}$ and $\mathcal{K}^c$ [the class of $\mathcal{L}$-structures which are not in $\mathcal{K}$] form elementary classes.*

Proof. Use the fact that the space of complete satisfiable $\mathcal{L}$-theories is a compact Hausdorff space. □

12. How large is an ultraproduct?

Proposition 12.1. *Suppose that $(M_i)_{i \in I}$ is a sequence of $\mathcal{L}$-structures such that $|M_i| = \aleph_0$ for each $i \in I$ and $I = \mathbb{N}$. Then*

$$\aleph_0 < \left| \prod_D M_i \right| \leq 2^{\aleph_0}$$

And thus, under the continuum hypothesis, we have the $\leq$ is equality.

Proof. We first show the lower bounded. Suppose that $\prod_D M_i$ is countable. Enumerate the elements of $\prod_D M_i$ via $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ where $\mathbf{a}_1 = [a_1^1, a_2^1, ...], \mathbf{a}_2 = [a_1^2, a_2^2, ...], \mathbf{a}_3 = [a_1^3, a_2^3, a_3^3, ...]$. Choose $[b_1, b_2, b_3, ...] \in \prod_{i \in I} M_i$ such that $b_1 \neq a_1^1$, $b_2 \neq a_2^1, a_2^2$, $b_3 \neq a_3^1, a_3^2, a_3^3$. Then, $\mathbf{b} = [b_1, b_2, b_3]_D$ is not in the list [by Los's theorem] and so we have a contradiction.

The upper bound follows from straightforward cardinal arithmetic. Notice,

$$\left| \prod_D M_i \right| \leq \left| \prod_{i \in \mathbb{N}} M_i \right| = |\mathbb{N}^{\mathbb{N}}| = |2^{\mathbb{N}}|.$$ □

We may ask how *rich* an ultraproduct is. To discuss this, we need to discuss the notion of saturation.

Definition 12.2. Suppose that M is an $\mathcal{L}$-structure and $A \subseteq M$. If $x_1, ..., x_n$ is a tuple of variables, then an $\mathcal{L}_{x_1,...,x_n}(A)$-formula is a formula which has free variables among $x_1, ..., x_n$ and possibly uses elements from A.

Definition 12.3. Suppose that M is an $\mathcal{L}$-structure and $A \subseteq M$. Then a partial n-type over A is a subset $\pi(x_1, ..., x_n)$ of $\mathcal{L}_{x_1,...,x_n}(A)$ such that every finite subset is satisfiable, i.e., for every $\pi_0(x_1, ..., x_n) \subseteq \pi(x_1, ..., x_n)$, there exists $b_1, ..., b_n \in M$ such that for every $\varphi(x_1, ..., x_n, \bar{a}) \in \pi_0(x_1, ..., x_n)$,

$$M \models \varphi(b_1, ..., b_n\bar{a}).$$

Definition 12.4. Suppose that M is an $\mathcal{L}$-structure and $A \subseteq M$. Then a complete n-type over A is a partial n-type $\pi(x_1, ..., x_n)$ over A such that for every formula $\theta(x_1, ..., x_n) \in \mathcal{L}_{x_1,...,x_n}(A)$, either $\theta(x_1, ..., x_n) \in \pi(x_1, ..., x_n)$ or $\neg\theta(x_1, ..., x_n) \in \pi(x_1, ..., x_n)$. We let $S_n(A)$ be the collection of complete n-types over A.

Example 12.5. Consider $M = (\mathbb{N}, <)$. Then $\{n < x : n \in \mathbb{N}\}$ is a partial 1-type over $\mathbb{N}$.

Example 12.6. Consider $M = (2^{<\omega}, \leq)$ where $\leq$ is the initial subset relation. Then any path through this tree gives a partial 1-type over $2^{<\omega}$. In other words, if $\gamma \in 2^{\omega}$, then $\{\gamma(n) \leq x : n \in \mathbb{N}\}$ is partial 1-type.

Fact 12.7. *Every partial n-type extends to a complete n-type.*

Definition 12.8. We say that an $\mathcal{L}$-structure M is κ-saturated if for every infinite cardinal $\lambda < \kappa$, if $A \subseteq M$ and $|A| < \lambda$, then every complete 1-type over A is realized in M, i.e., if $p \in S_1(A)$ then there exists some $a \in M$ such that for every $\theta(x) \in p$, $M \models \theta(a)$.

Theorem 12.9. *Suppose that $(M_i)_{i\in\mathbb{N}}$ is a sequence of $\mathcal{L}$-structures and D is a non-principal ultrafilter on $\mathbb{N}$. Then $\prod_D M_i$ is $\aleph_1$-saturated.*

Proof. Let $M = \prod_D M_i$ and let $C \subseteq M$ such that $|C| = \aleph_0$. Let $p \in S_1(C)$. We want to show that there exists some $d \in M$ such that $d \models p$. Enumerate the elements of p, $\langle \phi_i(x, \mathbf{a}_i^1, ..., \mathbf{a}_i^{m_i}) : i \in \mathbb{N}\rangle$. Let $\psi_i(x, \bar{\mathbf{a}}_i) = \bigwedge_{j\leq i} \phi_i(x, \mathbf{a}_j^1, ..., \mathbf{a}_j^{m_j})$. Since $p \in S_1(C)$, p is finitely satisfiable in M. Thus $M \models \exists x\psi_i(x, \bar{\mathbf{a}}_i)$. By Los's theorem, for each natural number i,

$$Y_i := \{t \in \mathbb{N} : M_t \models \exists x\psi_i(x, \bar{a}[t])\} \in D.$$

where $\bar{a}[t] = (a_1^1[t], ..., a_1^{m_1}[t], a_2^1[t], ..., a_i^{m_i}[t])$ and for each $\ell \leq i$ and $j \leq \ell$, $(a_\ell^j[t])_D = \mathbf{a}_\ell^j$. We now let $X_0 = \mathbb{N}$ and let $X_n = \{t \in \mathbb{N} : t \geq n\} \cap Y_n$. Note that $X_n \in D$. We can now build our sequence of points in $\prod_{i\in\mathbb{N}} M_i$ such that $[d[t]]_D \models p$. For each $t \in \mathbb{N}$, we $n(t)$ be the greatest natural number such that $t \in X_{n(t)}$. We define $d[t]$ as follows:

(1) If $n(t) = 0$, then choose $d[t] \in M_t$ to be anything.
(2) If $n(t) > 0$, choose $d[t] \in M_t$ such that $M_i \models \psi_{n(t)}(d[t])$.

We claim that for each $n \in \mathbb{N}$, $\prod_D M_i \models \phi_n([d[t]]_D)$. This follows from the obsercation that $X_n \subseteq \{t \in \mathbb{N} : M_t \models \phi_n(d[t])\}$. □

Remark 12.10. This statement above is true in a more general setting. Let I be an infinite indexing set. Let D be an ultrafilter on I. We say that D is regular if there exists a family of sets $(Z_n)_{n\in\mathbb{N}}$ from D such that $\bigcap Z_n = \emptyset$.

Proposition 12.11. *Suppose that I is an indexing set, D is a regular ultrafilter on I, and $(M_i)_{i\in I}$ is an indexed family of $\mathcal{L}$-structures. Then $\prod_D M_i$ is $\aleph_1$-saturated.*

12.1. Choosing ultrafilters.

It should be clear that if I take a sequence of $\mathcal{L}$-structures $(M_i)_{i\in I}$ and two ultrafilters D_1 and D_2 on I, it is possible that

$$\prod_{D_1} M_i \not\equiv \prod_{D_2} M_i.$$

Proposition 12.12. *Suppose that $\mathcal{L}$ is a countable language. If T is pseudofinite then there exists a sequence of finite structures $(M_i)_{i\in\mathbb{N}}$ such that for any non-principal ultrafilter D on $\mathcal{N}$ $\prod_D M_i \models T$.*

Proof. Enumerate $T = \varphi_1, \varphi_2, ...$ Let $\psi_n = \bigwedge_{j\leq n} \varphi_j$. For each j, find some M_j such that $M_j \models \psi_j$. Then, regardless of the choice of ultrafilter, we have that $\prod_D M_i \models T$ □

Can we give an explicit example of finite graphs such that any non-principal ultraproduct of this sequence results in the copy of the random graph? Yes.

Definition 12.13. Let $q = p^n$ be a prime power such that $q \equiv 1 \mod 4$. We define the Paley graph P_q to be the graph with set of vertices $V = \mathbb{F}_q$ - the unique finite field of cardinality q, and we say that two vertices a and b are connected if and only if $\exists z$ such that $(a-b) = z^2$ [and $a \neq b$].

Fact 12.14. *Let I be a non-principal ultrafilter on the set of indices $\{q : q$ is a prime power; $q \equiv 1 \mod 4\}$. Then for any non-principal ultrafilter D on I, we have that $\prod_D P_q$ is a model of the random graph.*

13. Almost Sure theories

Definition 13.1. Let $\mathcal{K}$ be a class of finite $\mathcal{L}$-structures closed under isomorphism. For each natural number n, we let $\mathcal{K}(n)$ denote the structures from $\mathcal{K}$ with underlying domain $\{1, ...n\}$. For each $\mathcal{L}$-sentence φ, we define

$$\mu_n(\varphi) = \frac{|\{M \in \mathcal{K}(n) : M \models \varphi\}|}{|\mathcal{K}(n)|}.$$

We let $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$. We let the almost sure theory of $\mathcal{K}$ be $T_{as}(\mathcal{K}) = \{\varphi : \mu(\varphi) = 1\}$.

Proposition 13.2. *Fix a class $\mathcal{K}$ which is closed under isomorphisms. Suppose that T is complete and T' is an axiomitization of T. If $T' \subseteq T_{as}(\mathcal{K})$ then $T = T_{as}(\mathcal{K})$.*

Proof. Fix $\varphi \in T$. By the compactness theorem, one can prove that there exists $\{\psi_1, ..., \psi_n\} \subseteq T'$ such that if $M \models \psi_1 \wedge ... \wedge \psi_n$ then $M \models \varphi$. The statement then follows from a straightfoward computation. □

Theorem 13.3. *Let $\mathcal{L}$ be a finite relational language. Let $\mathcal{K}$ be the class of all finite $\mathcal{L}$-structures. Then $T_{as}(\mathcal{K})$ is complete. In particular, $T_{as}(\mathcal{K}) = Th_{\mathcal{L}}(M_{\mathcal{K}})$ where $M_{\mathcal{K}}$ is the Fraïssé limit of said class.*

Proof. Recall that $Th_{\mathcal{L}}(M_{\mathcal{K}})$ can be characterized by one-point extension axioms. So, by Proposition 13.2, it suffices to prove that every one point extension axiom is in $T_{as}(\mathcal{K})$. Suppose that $\sigma(x_1, ..., x_n)$ is a complete quantifier free m-type and $\tau(x_1, ..., x_n, y)$ is a complete quantifier free $(m+1)$-type such that τ extends σ. We want to show that the following sentence

$$\varphi_{\sigma,\tau} := \forall x_1, ..., x_m \left(\left(\left(\left(\bigwedge_{i \neq j} x_i \neq x_j \right) \wedge \sigma(x_1, ..., x_n) \to \exists y \left(\left(\bigwedge_{i=1}^{m} y \neq x_i \right) \wedge \tau(x_1, ..., x_n, y) \right) \right) \right) \right).$$

is in $T_{as}(\mathcal{K})$. For simplicity, we let $\mathcal{L} = \{R\}$, a single binary relation. Now consider the following sequence of computations:

$$\begin{aligned}
\mu_n(\neg\varphi_{\sigma,\tau}) &= \mu_n\left(\exists x_1, ..., x_m\left(\left(\left(\bigwedge_{i\neq j} x_i \neq x_j\right) \wedge \underbrace{\sigma(\bar{x}) \wedge \forall y\left(\left(\bigwedge_{i=1}^{m} y \neq x_i\right) \wedge \tau(x_1, ..., x_n, y)\right)}_{\theta(\bar{x})}\right)\right)\right) \\
&= \mu_n(\exists x_1, ..., x_m\left(\bigwedge_{i\neq j} x_i \neq x_j \wedge \theta(\bar{x})\right)) \\
&= \sum_{\text{pairwise distinct } (a_1,...,a_m) \text{ from } \{1,...n\}} \mu(\theta(a_1, ..., a_m)) \\
&= \sum_{\text{pairwise distinct } (a_1,...,a_m) \text{ from } \{1,...n\}} \mu(\theta(1, ..., m)) \\
&= n(n-1)...(n-m+1)\mu(\theta(1, ..., m)) \\
&\leq n^m \mu(\theta(1, ..., m)) \\
&\leq n^m \mu_n\left(\forall y\left(\left(\bigwedge_{i=1}^{m} y \neq i\right) \rightarrow \psi(y)\right)\right) \\
&\leq n^m \prod_{j=m+1}^{n} \mu_n(\psi(j)) = n^m(1 - \frac{1}{2^{2m+1}})^{n-m} = n^m k^{n-m} \rightarrow 0.
\end{aligned}$$

where in the above computation, $\psi(y)$ is defined as follows: $\psi(y) = \neg(A_1 \wedge ... \wedge A_{2m+1})$ where for each $i \leq m$;

- $A_{2i-1} = R(i, y)$ if $R(x_i, y)$ is in $\tau(\bar{x}, y)$ and $\neg R(i, y)$ otherwise.
- $A_{2i} = R(y, i)$ if $R(y, x_i)$ is in $\tau(\bar{x}, y)$ and $\neg R(y, i)$ otherwise.
- $A_{2m+1} = R(i, i)$ if $R(y, y)$ is in $\tau(\bar{x}, y)$ and $\neg R(i, i))$ otherwise. □

Example 13.4. Let $\mathcal{K}_3$ be the class of all finite triangle free graphs and $\mathcal{K}_B$ be the class of all finite bipartite graphs. Then $\mathcal{K}_3$ and $\mathcal{K}_B$ are both elementary classes. We remark that $T_{as}(\mathcal{K}_3)$ is not equal to the theory of the Fraïssé of $\mathcal{K}_3$. This is because almost all triangle free graphs of some fixed size are bipartite. In particular, we have that $T_{as}(\mathcal{K}_3)$ says, "There are no 5 cycles" while the the Fraïssé limit admits a 5 cycle.

14. Graphon sampling

Definition 14.1. A graphon is a symmetric measurable function W from $[0, 1]^2$ to $[0, 1]$, i.e., $W(x, y) = W(y, x)$.

Definition 14.2. Let H be a graph on $\{1, ..., n\}$. Then the graph formula corresponding to H is precisely

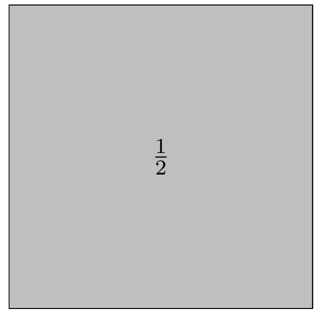
$$\varphi_H(x_1, ..., x_n) := \bigwedge_{\substack{i,j\in[n] \\ H\models R(i,j)}} R(x_i, x_j) \wedge \bigwedge_{\substack{i,j\in[n] \\ H\models\neg R(i,j)}} \neg R(x_i, x_j).$$

A graphon can help one build a random graph. For example, if H is a graph on $\{1, ..., n\}$ and we want to determine if we randomly construct H after n-step, we

(1) Randomly sample points $r_1, ..., r_n$ from the unit interval.
(2) For each pair (r_i, r_j), we flip a $W(r_i, r_j)$ weighted coin to determine if there is an edge.
(3) Check is the graph is isomorphic to H.

By *summing* up all the instances, we construct a measure on the space of graphs on $\mathbb{N}$.

Example 14.3. Consider the constant graphon given by $W(x, y) = 1/2$.

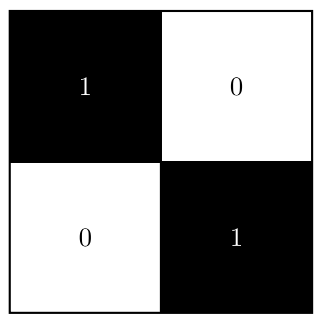


What is the probability that three sampled points form a triangle? Alternatively formulation: What is $G(\mathbb{N}, W)([\![R(1,2) \wedge R(2,3) \wedge R(1,3)]\!])$?

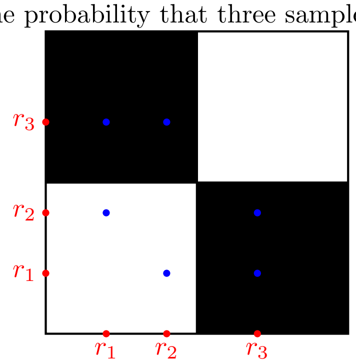
Answer 1: Compute the integral. $\int_{(t_1,t_2,t_3)\in[0,1]^3} \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} dL^3 = 1/8$.

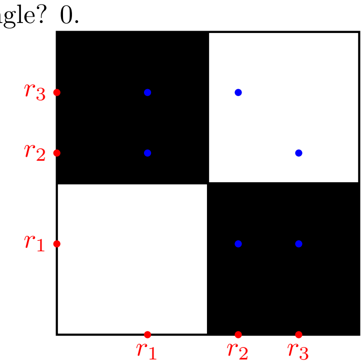
Answer 2: Reason probabilistically; if we randomly select r_1, r_2, r_3 from $[0,1]$, then the $W(r_i, r_j)$-weighted coin-flip is just a normal 50-50 coin-flip (since $W(r_i, r_j)$ is always $1/2$). Hence the probability that we have an edge between any two points is $1/2$. Since these coin-flips are independent, we arrive at $1/2 \cdot 1/2 \cdot 1/2 = 1/8$.

Example 14.4. Consider the following graphon W:



What is the probability that three sampled points form a triangle? 0.





Out of any three sampled points, at least two will be in either $[0, 1/2)$ or $(1/2, 1]$. These two points will not be connected by an edge. The configurations above show there are no triangles in either case.

Definition 14.5. Let $\mathcal{G}$ be the collection of graphs with underlying universe $\mathbb{N}$. Then $\mathcal{G}$ is a closed subspace of $\mathrm{Str}_{\mathcal{L}}$ [see next section for formal definition]. The processes of graphon sampling described above if equivalent to giving a measure on this particular space. Given a graphon W, and a graph H on $\{1, ..., n\}$, we have that

$$G(\mathbb{N}, W)([\![\varphi_H(i_1, ..., i_n)]\!] = \int_{(t_1,...,t_n)\in[0,1]^n} \prod_{1\le \ell < j \le n} W^{H(\ell,j)}(t_\ell, t_j) dL^n$$

where $W^{H(\ell,j)}(t_\ell, t_j) = W(t_\ell, t_j)$ is $H \models R(\ell, j)$ and $W^{H(\ell,j)}(t_\ell, t_j) = 1 - W(t_\ell, t_j)$ if $H \models \neg R(\ell, j)$. So, the probability that we sample a certain graph $H(x_1, ..., x_n)$ at step n is precisely $G(\mathbb{N}, W)([\![\varphi_H(1, ..., n)]\!])$.

Fact 14.6. *The following are equivalent: Let H be a coutnable graph.*

1. *There exists an invariant measure λ on $\mathcal{G}$ which concentrates on $H^{\cong}$ where $H^{\cong} := \{M \in \mathcal{G} : G \cong H\}$.*
2. *There exists a graphon W such that $G(\mathbb{N}, W)$ concentrates on $H^{\cong}$.*

The take away: If we can figure out when there exists an invariant measure which concentrates on the isomorphism class of a graph, then we are able to sample the graph – i.e., there exists a probabilistic construction of said graph.

14.1. **Formalism.** The purpose of this section is to prove a result by Ackerman, Freer, and Patel, which states precisely when there exists an invariant measure concentrating on a class of $\mathcal{L}$-structures. The main theorem of this section will be the following: Let A be a countable $\mathcal{L}$-structure. Then the following are equivalent:

1. There exists a measure μ on the space of labeled $\mathcal{L}$-structures, $\mathrm{Str}_{\mathcal{L}}$, such that μ is invariant [under the action of $Sym(\mathbb{N})$] and μ concentrates on $A^{\cong} := \{B \in \mathrm{Str}_{\mathcal{L}} : B \cong A\}$.
2. A has trivial group theoretic definable closure. If we let $\bar{a} = a_1, ..., a_n \in A$, then we define the group theoretic definable closure of A to be $dcl(\bar{a}) := \{b \in A$: for every automorphism σ of A which fixes $a_1, ..., a_n$ pointwise, σ also fixes $b\}$. We say that A has trivial group theoretic definable closure if for every $\bar{a} = a_1, ..., a_n \in A$, $dcl(\bar{a}) = \bar{a}$

Definition 14.7. Fix a countable language $\mathcal{L}$. We let $\mathrm{Str}_{\mathcal{L}}$ denote the space of labeled $\mathcal{L}$-structures. In other words, $\mathrm{Str}_{\mathcal{L}}$ is the collection of all $\mathcal{L}$-structures with underlying universe $\mathbb{N}$.

Definition 14.8. An $L_{\omega_1,\omega}(\mathcal{L})$-formula is of the following:

1. All atomic $\mathcal{L}$-formulas are $L_{\omega_1,\omega}(\mathcal{L})$-formulas.
2. If $\varphi(\bar{x})$ is an $L_{\omega_1,\omega}(\mathcal{L})$-formula then $\exists x \varphi(\bar{x})$ and $\neg\varphi(\bar{x})$ are $L_{\omega_1,\omega}(\mathcal{L})$-formulas.
3. If $\{\varphi_i(\bar{x})\}_{i\in I}$ is a countable family of $L_{\omega_1,\omega}(\mathcal{L})$-formulas, then $\bigwedge_{i\in I} \varphi_i(\bar{x})$ is an $L_{\omega_1,\omega}(\mathcal{L})$-formula provided that if has only finitely many free variables.

Fact 14.9 (Scott)**.** *Given an $\mathcal{L}$-structure A, there exists a single sentence φ_A in $L_{\omega_1,\omega}(\mathcal{L})$ such that for any countable $\mathcal{L}$-structure B, $B \models \varphi_A$ if and only if $B \models A$.*

Remark 14.10. $\bigvee = \neg \bigwedge \neg$ and $\forall = \neg\exists\neg$.

Definition 14.11. Given an $L_{\omega_1,\omega}(\mathcal{L})$-formula $\varphi(x_1, ..., x_n)$ and natural numbers $i_1, ..., i_n$, we let

$$[\![\varphi(i_1, ..., i_n)]\!] = \{M \in \mathrm{Str}_{\mathcal{L}} : M \models \varphi(i_1, ..., i_n)\}$$

Fact 14.12. *For any $L_{\omega_1,\omega}(\mathcal{L})$-formula $\theta(x_1, ..., x_n)$ and natural numbers $i_1, ..., i_n$, the set $[\![\varphi(i_1, ..., i_n)]\!]$ is a Borel subset of $\mathrm{Str}_{\mathcal{L}}$. Thus, by Fact 14.9, for any countable $\mathcal{L}$-structure A, we have that*

$$\{M \in \mathrm{Str}_{\mathcal{L}} : M \cong A\}$$

is a Borel subset of $\mathrm{Str}_{\mathcal{L}}$.

Fact 14.13. *$\mathrm{Str}_{\mathcal{L}}$ is a compact Hausdorff topological space with a subbasis of open sets given by*

$$\{[\![R^{\epsilon}(i_1, ..., i_{\mathrm{ar}(R)})]\!] : R \in \mathcal{L}; i_1, ..., i_{\mathrm{ar}(R)} \in \mathbb{N}; \epsilon \in \{\pm 1\}\},$$

where $R^1(-) = R(-)$ and $R^{-1}(-) = \neg R(-)$.

Remark 14.14. Since $\mathrm{Str}_{\mathcal{L}}$ is a compact Hausdorff space, we can consider Borel prob. measures on it.

15. Hrushovski Construction

Definition 15.1 (Pre-geometry)**.** A pregeometry is a set X together with a map $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ is the following properties:

(1) If $A \subseteq \mathcal{P}(X)$, then $A \subseteq cl(A)$.
(2) If $A \subseteq B$ then $cl(A) \subseteq cl(B)$.
(3) $cl(cl(A)) = cl(A)$
(4) If $a \in cl(A \cup \{b\}) \backslash cl(A)$, then $b \in cl(A \cup \{a\})$.
(5) $cl(A) = \bigcup \{cl(F) : F \subseteq_{finite} A\}$.

We say that a set $B \subseteq X$ is closed if $cl(B) = B$.

Definition 15.2. Let (X, cl) be a pregeometry. Supposed that $A \subseteq B \subseteq X$ and B is closed. We say that

(1) A is independent if for any $a \in A$, $a \notin cl(A \backslash \{a\})$.
(2) We say that A is a basis for B is A is independent and $cl(A) = B$.

Example 15.3. The span operation over a vector space; the algebraic closure operation over an algebraically closed field.

Definition 15.4. Let M be a first order $\mathcal{L}$-structure. We say that M is strongly minimal if for ever $\mathcal{L}$-formula $\varphi(x; \bar{y})$ there exists a natural number n_φ such that for every $\bar{b} \in M^{|\bar{y}|}$, either

$$|\{a \in M : M \models \varphi(a, \bar{b})\}| \leq n_\varphi \text{ or } |\{a \in M : M \models \neg\varphi(a, \bar{b})\}| \leq n_\varphi.$$

Definition 15.5. Fix an $\mathcal{L}$-structure M and suppose that A is a subset of M. The algebraic closure of A, denoted $acl(A)$, is the collection of points $b \in M$ such that there exists an $\mathcal{L}$-formula $\varphi(x, \bar{y})$ and parameters $\bar{a} \in A^{|\bar{y}|}$ such that $M \models \varphi(b, \bar{a})$ and $|\{b \in M : M \models \varphi(b, \bar{a})\}| < \aleph_0$.

Fact 15.6. *If M is strongly minimal, then (M, acl) forms a pregeometry.*

Zilber conjectured that essentially there are only three kinds of pregeometries: Ones that look trivial, ones that look like groups, and ones that interpret algebraically closed fields. Hrushovski gave a counterexample to this conjecture by variant construction on the Fraïssé limit.

15.1. **Hrushovski Construction.**

Let $\mathcal{L}$ be a countable relational language so that there rare only finitely many relations of each arity. We write $\mathcal{L} = \{R_i : i < \omega\}$. Choose a list of non-negative real numbers $(\alpha_i : i < \omega)$; each α_i is the *weight* of the relation R_i. Suppose that M is an $\mathcal{L}$-structure and suppose the arity of R_i is r_i. We let $R_i(M) = \{\bar{a} \in M^{r_i} : M \models R(\bar{a})\}$. Let A be a finite $\mathcal{L}$-structure. We define the predimension function as follows:

$$\delta(A) = |A| - \sum_{i<\omega} \alpha_i |R_i(A)|.$$

Throughout, we let $\mathcal{K}$ be the class of all finite $\mathcal{L}$-structures such that the relations R_i hold only on distinct elements for all $i < \omega$. This implies that there are only finitely many structures up to isomorphism of each finite cardinality. We let

$$\mathcal{K}_0 = \{A \in \mathcal{K} : \delta(B) > 0 \text{ for all non-empty } B \subseteq A\}.$$

Definition 15.7. Let B be an $\mathcal{L}$-structure. Fix $A_1, A_2 \subseteq B$ and $A_0 = A_1 \cap A_2$. Then A_1 and A_2 are *freely amalgamated* over A_0 if the only edges in $A_1 \cup A_2$ which are not edges in A_1 or A_2 have weight 0.

If A_1 and A_2 are freely amalgamated over A_0, the amalgam is *canonical* if there are no edges of $A_1 \cup A_2$ which are not edges of A_1 or A_2. We denote the canonical free amalgam of A_1 and A_2 over A_0 by $A_1 \otimes_{A_0} A_2$.

Remark 15.8. If A_1, A_2 are (canonically) freely amalgamated over $A_0 = A_1 \cap A_2$ and $B \subseteq A_1 \cup A_2$; we may put $B_i = B \cap A_i$ $i = 0, 1, 2$. Then B_1 and B_2 are (canonically) freely amalgamated over their intersection B_0.

Definition 15.9. Suppose that $A_0 \subseteq A_1$, $A'_0 \subseteq A_2$ such that $A_0 \cong A'_0$. Then one can obviously construct the free amalgam of A_1 and A_2 over $A_0 \cong A'_0$. We denote the amalgam as $A_1 \otimes_{A_0} A_2$.

Lemma 15.10. *Let $B \in \mathcal{K}$ and $A_1, A_2 \subseteq B$. If $A_0 = A_1 \cap A_2$, then*
$$\delta(A_1 \cup A_2) + \delta(A_0) \leq \delta(A_1) + \delta(A_2).$$
Equality holds if and only if A_1 and A_2 are freely amalgamated over A_0.

Definition 15.11 (Strong substructure)**.** If $B \in \mathcal{K}$, $A \subseteq B$, we write that $A \leq B$ if for every A' such that $A \subset A' \subseteq B$, then $\delta(A') > \delta(A)$. If $\sigma : A \to B$ is an embedding, we say that it is closed/strong if $\sigma(A) \leq B$.

The next three results are homework:

Lemma 15.12. *Let $B \in \mathcal{K}$ and $A \leq B$. If $B' \subseteq B$, then $A \cap B' \leq B'$.*

Lemma 15.13. *If $C \in \mathcal{K}$ and $A \leq B \leq C$ then $A \leq C$.*

Corollary 15.14. *If $B \in \mathcal{K}$ and $A \subseteq B$, then there is a unique minimal superstructure D of A in B.*

Lemma 15.15. *Let $B \in \mathcal{K}$, A_1, A_2 be substructures of B which are in $\mathcal{K}_0$. If A_1 and A_2 are freely amalgamated over their intersection $A_0 = A_1 \cap A_2$ and $A_0 \leq A_1$, then $A_1 \cup A_2 \in \mathcal{K}_0$ and $A_2 \leq A_1 \cup A_2$.*

Proof. First we show that $A_2 \leq A_1 \cup A_2$. Choose some D such that $A_0 \subset D \subseteq A_1 \cup A_2$. Let $D_1 = D \cap A_1$. Then $A_0 \subset D_1 \subseteq A_1 \implies \delta(A_0) < \delta(D_1)$ since $A_0 \leq A_1$. Moreover, D is a free amalgam of A_2 and D_1 over A_0. Thus,
$$\delta(D) = \delta(D_1) + \delta(A_2) - \delta(A_0) > \delta(A_2)$$
By the above, we know that $\delta(D_1) - \delta(A_0) > 0$ and so the inequality above holds.

Now we prove that $A_1 \cup A_2 \in \mathcal{K}_0$. Choose $C \subseteq A_1 \cup A_2$ and suppose that C is non-empty. We have two cases:

(1) $C \subseteq A_1$. We are done since $A_1 \in \mathcal{K}_0$.
(2) $C \cap A_2$ is non-empty. Then by Lemma 15.12, $C \cap A_2 \leq C$. Hence,
$$0 < \delta(C \cap A_2) \leq \delta(C).$$
where, the first inequality follows from the fact that $C \cap A_2 \subseteq A_2$ and $A_2 \in \mathcal{K}_0$. The second inequality follows from the fact that $C \cap A_2$ is a strong substructure of C.

□

We now expand the families of structures we are interested in:

Definition 15.16. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone increasing unbounded function such that $f(0) = 0$ and $\emptyset \in \mathcal{K}_f$. We put
$$\mathcal{K}_f := \{A \in \mathcal{K} : \delta(B) \geq f(|B|) \text{ for all } B \subseteq A\}.$$

Note that $\mathcal{K}_f \subseteq \mathcal{K}$.

Definition 15.17. If M is an infinite $\mathcal{L}$-structure where all of the finite substructures are in $\mathcal{K}$ and $A \in \mathcal{K}$, we write $A \leq M$ if for every finite A' such that $A \subseteq A' \subseteq M$, we have that $A \leq A'$.

Definition 15.18. We say that $\mathcal{K}_f$ is closed under free amalgamation if whenever $A_0 \leq A_1, A_2 \in \mathcal{K}_f$, then $A_1 \otimes_{A_0} A_2 \in \mathcal{K}_f$.

Definition 15.19. A countable $\mathcal{L}$-structure M is a generic model of $\mathcal{K}_f$ if the age of M is $\mathcal{K}_f$ and whenever $A \leq M$ and $A \leq B \in \mathcal{K}_f$, then there is a closed embedding $\sigma : B \to M$ over A.

Theorem 15.20. *Suppose that $\mathcal{K}_f$ is non-trivial and closed under free amalgamation. Then there is a generic model for $\mathcal{K}_f$.*

Proof. Similar to the proof of Fraïssé's theorem except instead of ranging all substructures and all embeddings, one restricts the construction to strong substructures and strong embeddings. Build the model in stages. Suppose at stage n we have constructed M_n. Enumerate all strong substructures $B_1, ..., B_{k_n}$ of M_n. Enumerate all pairs $(D_\ell, C_\ell)_{\ell<s}$ where $D_l \cong B_t$ for $t \leq k_n$, $C \cong A_j$ for $j \leq n+1$ and $D_l \leq C_l$. Then we let $X_0 = M_n$, $X_{j+1} = X_j \otimes_{D_\ell} C_\ell$, and $M_{n+1} = X_s$. Then $M = \bigcup_{n<\omega} M_n$. One checks that M satisfies the conconclusion of the statement. □

Lemma 15.21. *Let M be a generic model of $\mathcal{K}_f$ where $\mathcal{K}_f$ is non-trivial and admits free amalgamation. Then there is a unique minimal finite set $B \leq M$ containing A, which we denote as $cl_M(A)$.*

Theorem 15.22. *Let M be a generic model of $\mathcal{K}_f$ where $\mathcal{K}_f$ is non-trivial and admits free amalgamation. Then for any finite set $A \subseteq M$, $cl_M(A) = acl(A)$.*